

MANIFOLDS CONTAINING AN AMPLE \mathbb{P}^1 -BUNDLE

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ABSTRACT. Sommese has conjectured a classification of smooth projective varieties X containing, as an ample divisor, a \mathbb{P}^d -bundle Y over a smooth variety Z . This conjecture is known if $d > 1$, if $\dim(X) \leq 4$, or if Z admits a finite morphism to an Abelian variety. We confirm the conjecture if the Picard rank $\rho(Z) = 1$, or if Z is not uniruled. In general we reduce the conjecture to a conjectural characterization of projective space: namely that if W is a smooth projective variety, \mathcal{E} is an ample vector bundle on W , and $\mathrm{Hom}(\mathcal{E}, T_W) \neq 0$, then $W \simeq \mathbb{P}^n$.

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1. INTRODUCTION

Beltrametti and Sommese give a conjectural classification of smooth projective varieties X containing a \mathbb{P}^d -bundle as an ample divisor [BS95, Conjecture 5.5.1]. The main goal of this paper is to prove this conjecture in the case where X has minimal Picard rank,

$$\rho(X) = 2.$$

Throughout the paper we work over \mathbb{C} ; the phrase “ \mathbb{P}^d -bundle,” will be taken to mean a \mathbb{P}^d -bundle locally trivial in the analytic topology.

The conjecture is:

Conjecture 1. *Let X be a smooth projective variety and $Y \subset X$ a smooth ample divisor. Suppose that $p : Y \rightarrow Z$ is a morphism exhibiting Y as a \mathbb{P}^d -bundle over a b -dimensional manifold Z . Then one of the following holds:*

- (1) $X \simeq \mathbb{P}^3$, $Y \simeq \mathbb{P}^1 \times \mathbb{P}^1$ is a smooth quadric, and p is one of the projections to \mathbb{P}^1 .
- (2) $X \simeq Q^3 \subset \mathbb{P}^4$ is a smooth quadric threefold, $Y \simeq \mathbb{P}^1 \times \mathbb{P}^1$ is a hyperplane section, and p is a projection to one of the factors.
- (3) $Y \simeq \mathbb{P}^1 \times \mathbb{P}^b$, $Z \simeq \mathbb{P}^b$, $p : Y \rightarrow Z$ is the projection to the second factor, and X is the projectivization of an ample vector bundle \mathcal{E} on \mathbb{P}^1 .
- (4) $X \simeq \mathbb{P}(\mathcal{E})$ for an ample vector bundle \mathcal{E} on Z , and $\mathcal{O}_X(Y) \simeq \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ (i.e. Y is a fiberwise hyperplane).

Sommese has proven the case where $d \geq 2$ (see e.g. [BS95, Theorem 5.5.2]). The cases where $d = 1$ and $b = 1, 2$ are due to the work of several authors (see

e.g. [BI09, Theorem 7.4] and the references therein). We prove the conjecture in the case where

$$\rho(Z) = 1,$$

(if $\dim(X) \geq 4$, this is equivalent to $\rho(X) = 2$) and in general reduce it to a plausible conjectural improvement of a result of Andreatta-Wisniewski [AW01], namely

Conjecture 2. *Let X be a smooth projective variety and \mathcal{E} an ample vector bundle on X . If*

$$\mathrm{Hom}(\mathcal{E}, T_X) \neq 0,$$

then $X \simeq \mathbb{P}^n$.

We also prove the conjecture in the case that Z is not uniruled.

Remark 3. By [AW01], the existence of a map $\mathcal{E} \rightarrow T_X$ of constant rank implies $X \simeq \mathbb{P}^n$; likewise, [AKP08, Corollary 4.3] proves the conjecture if $\rho(X) = 1$. One may also use the methods of [AKP08, Section 4] to prove the conjecture if there exists a map $\mathcal{E} \rightarrow T_X$ generically of maximal rank.

The idea of our argument is to show (via an analysis of the deformation theory of the map $p : Y \rightarrow Z$) that either p extends to a map $\hat{p} : X \rightarrow Z$ (using results of [Lit16]), or there is an ample vector bundle \mathcal{E} on Z and a map $\mathcal{E} \rightarrow T_Z$. In the former case, we are done by work of Sommese; in the latter case, we may apply Conjecture 2 to proceed.

Acknowledgments. This note owes a debt to conversations with Tommaso de Fernex, Paltin Ionescu, and Jason Starr. It was written with support from an NSF Postdoctoral Fellowship.

2. THE PROOF

We first show:

Lemma 4. *As before, let X be a smooth projective variety, $Y \subset X$ a smooth ample divisor, and $p : Y \rightarrow Z$ a \mathbb{P}^1 -bundle. Let \hat{Y} be the formal scheme obtained by completing X at Y . If $p : Y \rightarrow Z$ does not extend to a morphism $\hat{p} : \hat{Y} \rightarrow Z$, then there exists an ample vector bundle \mathcal{E} on Z and a non-zero morphism $\mathcal{E} \rightarrow T_Z$.*

Proof. Let \mathcal{I}_Y be the ideal sheaf of Y , and let Y_n be the subscheme of X cut out by \mathcal{I}_Y^n . Then the obstruction to extending a map

$$p_n : Y_n \rightarrow Z$$

to a map

$$Y_{n+1} \rightarrow Z$$

lies in

$$\mathrm{Ext}^1(p^*\Omega_Z^1, \mathcal{I}_Y^n/\mathcal{I}_Y^{n+1}) = H^1(Y, p^*T_Z \otimes \mathcal{I}_Y^n/\mathcal{I}_Y^{n+1}).$$

This last is equal to

$$H^1(\mathbf{R}p_*(p^*T_Z \otimes \mathcal{I}_Y^n/\mathcal{I}_Y^{n+1}))$$

which is the same as

$$H^1(T_Z \otimes \mathbf{R}p_*\mathcal{I}_Y^n/\mathcal{I}_Y^{n+1})$$

by the projection formula. As $\mathcal{I}_Y^n/\mathcal{I}_Y^{n+1}$ is anti-ample, this last equals

$$H^0(T_Z \otimes \mathbf{R}^1p_*\mathcal{I}_Y^n/\mathcal{I}_Y^{n+1}).$$

Applying Serre duality, we see that this is the same as

$$H^0(T_Z \otimes (p_*(\mathcal{O}(nY)|_Y \otimes \omega_{Y/Z}))^\vee) = \text{Hom}(p_*(\omega_{Y/Z} \otimes \mathcal{O}(nY)|_Y), T_Z).$$

But by [MT07, Theorem 1.2],

$$p_*(\omega_{Y/Z} \otimes \mathcal{O}(nY)|_Y)$$

is either zero or ample. Thus either the problem of extending p to \widehat{Y} is unobstructed, or the obstruction is a non-zero map from an ample vector bundle \mathcal{E} on Z to T_Z , as desired. \square

We will also require:

Lemma 5. *Let X be a smooth projective variety of dimension at least 3, and $Y \subset X$ an ample divisor. Let Z be a smooth variety with $\dim(Z) < \dim(Y)$. Then the restriction map*

$$\text{Hom}(X, Z) \rightarrow \text{Hom}(\widehat{Y}, Z)$$

is a bijection. Here \widehat{Y} is, as before, the formal scheme obtained by completing X at Y .

Proof. This is a combination of two results from [Lit16]. First, by [Lit16, Corollary 2.10], applied to the projection $X \times Z \rightarrow X$, a map $p : \widehat{Y} \rightarrow Z$ extends uniquely to some Zariski-open neighborhood U of Y . Second, by [Lit16, Corollary 3.3], this rational map to Y is in fact regular. \square

Corollary 6. *Let X, Y, Z, p be as in Conjecture 1. Suppose that either*

- (1) *Z is not uniruled, or*
- (2) *$\rho(Z) = 1$ (equivalently, $\rho(X) = \rho(Y) = 2$).*

Then Conjecture 1 is true for X, Y, Z, p .

Proof. Without loss of generality, p has relative dimension 1 (i.e. it exhibits Y as a \mathbb{P}^1 -bundle over Z) as the case of relative dimension greater than 1 is already known [BS95, Theorem 5.5.2]. We may also assume $\dim(Z) > 2$, as again, if $\dim(Z) \leq 2$, the result is already known [BI09, Theorem 7.4].

By Lemma 4, either p extends to a map $\hat{p} : \widehat{Y} \rightarrow Z$ or T_Z contains an ample subsheaf, namely the image of \mathcal{E} from Lemma 4. In the former case, the map p extends to a map $\tilde{p} : X \rightarrow Z$ by Lemma 5 and we are done by [BI09, Theorem 5.5(ii)] (in particular, we are in case (4) of the conjecture). In the latter case, we consider the situations

- (1) Z not uniruled, or
- (2) $\rho(Z) = 1$

separately.

- (1) Suppose Z is not uniruled. Then T_Z contains no ample subsheaves by a result of Miyaoka (see e.g. [Kol96, IV.1.16]), so we have a contradiction.
- (2) Alternately, suppose $\rho(Z) = 1$. Then as T_Z contains an ample subsheaf, by [AKP08, Corollary 4.3], $Z \simeq \mathbb{P}^n$. By [FSS87, Theorem 2.1] we conclude the result, namely that we are in case (3) of the conjecture. \square

Corollary 7. *Suppose that Conjecture 2 is true. Then Conjecture 1 holds as well.*

Proof. This is the same argument as in the $\rho(Z) = 1$ case above, replacing the reference to [AKP08] with Conjecture 2. \square

REFERENCES

- [AKP08] Marian Aprodu, Stefan Kebekus, and Thomas Peternell. Galois coverings and endomorphisms of projective varieties. *Math. Z.*, 260(2):431–449, 2008. 2, 3, 4
- [AW01] Marco Andreatta and Jarosław A. Wiśniewski. On manifolds whose tangent bundle contains an ample subbundle. *Invent. Math.*, 146(1):209–217, 2001. 2
- [BI09] Mauro C. Beltrametti and Paltin Ionescu. A view on extending morphisms from ample divisors. In *Interactions of classical and numerical algebraic geometry*, volume 496 of *Contemp. Math.*, pages 71–110. Amer. Math. Soc., Providence, RI, 2009. 2, 3
- [BS95] Mauro C. Beltrametti and Andrew J. Sommese. *The adjunction theory of complex projective varieties*, volume 16 of *de Gruyter Expositions in Mathematics*. Walter de Gruyter & Co., Berlin, 1995. 1, 3
- [FSS87] Maria Lucia Fania, Ei-ichi Sato, and Andrew John Sommese. On the structure of 4-folds with a hyperplane section which is a \mathbf{p}^1 bundle over a surface that fibres over a curve. *Nagoya Math. J.*, 108:1–14, 1987. 3
- [Kol96] János Kollár. *Rational curves on algebraic varieties*, volume 32 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 1996. 3
- [Lit16] D. Litt. Non-Abelian Lefschetz Hyperplane Theorems. *ArXiv e-prints*, January 2016. 2, 3
- [MT07] Christophe Mourougane and Shigeharu Takayama. Hodge metrics and positivity of direct images. *J. Reine Angew. Math.*, 606:167–178, 2007. 3